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3.  $b : b' :: 4 : 3$ . 3rd condition.

4.  $v : v' :: 21 : 21$ , multiplying and reducing, and remembering that the value  $\propto l.b.q.$

5. Also  $v - v' = \$500$ . Whence,

6.  $v = \$10500$ . From (4) and (5),

7.  $v' = \$10000$ .

This problem was also solved by *B. F. SINE, NELSON S. RORAY, P. S. BERG, F. M. McGAW, J. C. CORBIN, COOPER D. SCHMITT, FREDERIC R. HONEY, H. C. WILKES, and G. B. M. ZERR.*

M. A. Gruber sent in a solution of Problem 70, Department of Arithmetic, too late for credit in last issue. His answer is 6.48 years.

## ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

## SOLUTIONS OF PROBLEMS.

68. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to  $n$  terms the series,  $n\cos\theta + (n-1)\cos2\theta + (n-2)\cos3\theta$ , etc.

[*Chrystal's Algebra.*]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

Let  $S = n\cos\theta + (n-1)\cos2\theta + (n-2)\cos3\theta + \dots$ . Also let  $S_s = \sin\theta + \sin2\theta + \sin3\theta + \dots$ , and  $S_c = \cos\theta + \cos2\theta + \cos3\theta + \dots$ .

$$\begin{aligned} S &= n[\cos\theta + \cos2\theta + \cos3\theta + \dots] - [\cos2\theta + 2\cos3\theta + \dots], \\ &= (n+1)[\cos\theta + \cos2\theta + \cos3\theta + \dots] - [\cos\theta + 2\cos2\theta + 3\cos3\theta + \dots], \\ &= (n+1)S_c - dS_s/d\theta. \end{aligned}$$

Now  $S_c = [\cos\{\frac{1}{2}(n+1)\theta\} \sin\frac{1}{2}(n\theta)] / \sin\frac{1}{2}\theta$ , and  $S_s = [\sin\{\frac{1}{2}(n+1)\theta\} \sin\frac{1}{2}(n\theta)] / \sin\frac{1}{2}\theta$ .

$$\therefore S = (n+1) \frac{\cos\{\frac{1}{2}(n+1)\theta\} \sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta} - \frac{d}{d\theta} \left[ \frac{\sin\frac{1}{2}(n+1)\theta \sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta} \right],$$

probably as compact a form as can be obtained.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let  $S =$  sum required,

$$2\sin\frac{1}{2}\theta \cos n\theta = \sin\left\{\theta + \frac{2n-1}{2}\theta\right\} - \sin\left\{\theta + \frac{2n-3}{2}\theta\right\}$$

$$4\sin\frac{1}{2}\theta\cos(n-1)\theta=2\sin\{\theta+\frac{2n-3}{2}\theta\}-2\sin\{\theta+\frac{2n-5}{2}\theta\}$$

$$6\sin\frac{1}{2}\theta\cos(n-2)\theta=3\sin\{\theta+\frac{2n-5}{2}\theta\}-3\sin\{\theta+\frac{2n-7}{2}\theta\}$$

$$8\sin\frac{1}{2}\theta\cos(n-3)\theta=4\sin\{\theta+\frac{2n-7}{2}\theta\}-4\sin\{\theta+\frac{2n-9}{2}\theta\}$$

.....

$$2n\sin\frac{1}{2}\theta\cos\theta=n\sin(\theta+\frac{1}{2}\theta)-n\sin(\theta-\frac{1}{2}\theta).$$

Adding we get

$$\begin{aligned} 2S\sin\frac{1}{2}\theta &= (\sin\frac{3}{2}\theta + \sin\frac{5}{2}\theta + \sin\frac{7}{2}\theta + \dots + \sin\frac{2n+1}{2}\theta) - n\sin\frac{1}{2}\theta, \\ &= [\sin(\frac{n+2}{2}\theta)\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta - n\sin\frac{1}{2}\theta. \end{aligned}$$

$$\therefore S = [\sin(\frac{n+2}{2}\theta)\sin\frac{1}{2}(n\theta) - n\sin^2\frac{1}{2}\theta]/2\sin^2\frac{1}{2}\theta.$$

The series in parenthesis above is summed in all trigonometries in the series,  $\sin\alpha + \sin(\alpha + \beta) +$ , etc., by making  $\alpha = \frac{3}{2}\theta$ ,  $\beta = \theta$ .

### III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The given series may be broken up into :

$$\begin{aligned} n[\cos\theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta] \\ - [\cos 2\theta + 2\cos 3\theta + 3\cos 4\theta + \dots + (n-1)\cos n\theta]. \end{aligned}$$

To sum the series  $\cos\theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta$ , we have

$$\sin\frac{1}{2}\theta - \sin\frac{3}{2}\theta = -2\cos\theta\sin\frac{1}{2}\theta.$$

$$\sin\frac{3}{2}\theta - \sin\frac{5}{2}\theta = -2\cos 2\theta\sin\frac{1}{2}\theta.$$

$$\sin\frac{5}{2}\theta - \sin\frac{7}{2}\theta = -2\cos 3\theta\sin\frac{1}{2}\theta.$$

.....

$$\sin\frac{1}{2}(2n-1)\theta - \sin\frac{1}{2}(2n+1)\theta = -2\cos n\theta\sin\frac{1}{2}\theta.$$

$$\text{Adding, we have, } \sin\frac{1}{2}\theta - \sin\frac{1}{2}(2n+1)\theta = -2\sin\frac{1}{2}\theta\Sigma(n\theta).$$

$$\therefore \Sigma(n\theta) = [\sin\frac{1}{2}(2n+1)\theta - \sin\frac{1}{2}\theta]/2\sin\frac{1}{2}\theta = [\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta]/\sin\frac{1}{2}\theta.$$

$$\therefore n\Sigma(n\theta) = [n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta]/\sin\frac{1}{2}\theta.$$

To sum the second part, we have,

$$x^2 + 2x^3 + 3x^4 + \dots + (n-1)x^n = [x^2 - nx^{n+1} + (n-1)x^{n+2}]/(1-x)^2.$$

Putting  $x = \cos\theta + i\sin\theta$ , and employing the formula  $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$ , we obtain after putting the real parts of both members equal, and making all necessary reductions, for the sum of the second series

$$= \frac{\cos\theta - n\cos n\theta + (n-1)\cos(n+1)\theta}{4\sin^2 \frac{1}{2}\theta};$$

so that the sum of the given series

$$= \frac{n\cos \frac{1}{2}(n+1)\theta \sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta} + \frac{\cos\theta - n\cos n\theta + (n-1)\cos(n+1)\theta}{4\sin^2 \frac{1}{2}\theta}.$$

To test this formula we must of course, leave the coefficient  $n$  of the first expression unchanged, while in all the other factors and terms which involve  $n$ ,  $n$  must be put successively  $= 1, 2, 3, 4$ , etc.

Also solved by *E. W. MORRELL*.

69. Proposed by *C. E. WHITE, A. M., Trafalgar, Indiana.*

Prove that  $x^n \pm x^{n-1} + x^{n-2} \pm \dots + (\pm 1)^{n-1}x + (\pm 1)^n = (x \mp 1)^n \pm A(x \mp 1)^{n-1} + B(x \mp 1)^{n-2} \pm \dots + (\pm 1)^n x$ , where  $A, B, C, \dots$  are the binomial coefficients of the  $(n+1)$ th order.

**I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.**

$$x^n \pm x^{n-1} + x^{n-2} + \dots, \text{ etc.} = (x^{n+1} - 1)/(x - 1) \dots \dots \dots (1),$$

$$\text{or } x^{n+1} + 1)/(x + 1) \dots \dots \dots (2), = \{[(x - 1) + 1]^{n+1} - 1\}/(x - 1), \text{ or}$$

$$\{[(x + 1) - 1]^{n+1} + 1\}/(x + 1), = (x - 1)^n + C_{n+1}^2(x - 1)^{n-1} + C_{n+1}^3(x - 1)^{n-2} + \dots,$$

$$\text{or } (x + 1)^n - C_{n+1}^2(x + 1)^{n-1} + C_{n+1}^3(x - 1)^{n-2} - \dots,$$

$$= (x \mp 1)^n \pm A(x \mp 1)^{n-1} + B(x \mp 1)^{n-2} + \dots$$

**II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.**

$$\text{Let } K = x^n \pm x^{n-1} + x^{n-2} \pm \dots + (\pm 1)^{n-1}x + (\pm 1)^n.$$

Put  $x = y \pm 1$ , expanding and observing that the sign of the last term of each expression is  $\pm$  if  $n$  is odd but  $+$  if  $n$  is even, we may write :

$$x^n = (y \pm 1)^n = y^n \pm n y^{n-1} + \frac{1}{2}[n(n-2)]y^{n-2} \pm \dots + (\pm 1)^{n-1}ny + (\pm 1)^n$$

$$\pm x^{n-1} = \pm (y \pm 1)^{n-1} = \pm y^{n-1} + (n-1)y^{n-2} \pm \dots + (\pm 1)^{n-1}(n-1)y + (\pm 1)^n$$

$$x^{n-2} = (y \pm 1)^{n-2} = \dots y^{n-2} \pm \dots + (\pm 1)^{n-1}(n-2)y + (\pm 1)^n$$

$$\text{etc } \dots \dots \dots \text{etc.}$$